Group Y Math-M-Addicts Entrance Exam 2014 – 2015

Student Name ____Grade in School ____________________

School Name ____Email _________________________________

Instructions: A perfect solution to each problem below is worth 10 points. A score of 40 or more will guarantee admission to group Y, though depending on the submitted tests, the cutoff for admission may be lower. **Students must work on the problems on their own and not receive any help; students suspected of cheating will be disqualified.** Good luck!

1. Using numbers 5, 6, 7, 8 and basic arithmetic operations (i.e., +, -, *, /, and parentheses), generate 24. Each number has to be used exactly once and combining numbers (e.g., 78 - 56) is not allowed.

2. Find all solutions to TAE + TEA = EAT; each letter corresponds to a digit; same letter corresponds to the same digit and different letters correspond to different digits.

3. Positive numbers p and $p^2 + 7$ are both prime. What are all possible values of p? Justify your answer.

4. A scientist gathered four math students. They were lined up so that each one could see the ones in front of them but not behind them. Each student had a hat placed on their head. So the student in the back could see the hats of the three students in front, but the student in front could not see any hats. *"There is a red hat, a white hat, a blue hat, and a hat that is a duplicate of one of those colors,"* the scientist said. Starting with the one in the back, each student was asked what color hat they were wearing. They were all able to give correct answers without guessing! Which two students were wearing the two hats of the same color? Explain your reasoning.

5. If 1 + 2 + 3 + 4 + ... + 300 = 45,150, compute 2 + 4 + 6 + 8 + ... + 300 without using a calculator. Explain your reasoning.

6. All numbers between 1 and 999 were written out into one very large number. How many 7's were used? Explain your reasoning.

Group Y Math-M-Addicts Entrance Exam 2015 – 2016

Instructions: A perfect solution to each problem below is worth 10 points. A score of 40 or more will guarantee admission to group S, though depending on the submitted tests, the cutoff for admission may be lower. Students must work on the problems on their own and not receive any help; students suspected of cheating will be disqualified. Please justify all your answers; partial credit may be given for incomplete solutions.

Good luck!

Student's Name: _______________________________

Problem 1. A book has 120 pages numbered 1,2,3,…, 120. How many times is the digit "9" used in page numbers?

Problem 2. A box contains 100 balls. They are colored red, white and blue. Emily is picking the balls from the box without looking. What is the smallest number of balls she can take to be absolutely sure there would be 20 balls of the same color among the ones picked?

Problem 3. There are two water bottles; one of them can hold 8 quarts of water and the other can hold 5 quarts. You are on the shore of a lake. How can you use these bottles to get exactly 7 quarts of water in the bigger bottle?

Problem 4. Show how the 1x1 cells of a square 6x6 can be colored in 3 colors so that each cell shares a side with exactly one cell of the other two colors. (So if the colors are red, white and blue, and you have a white cell somewhere, it must have exactly one blue and one red neighbor.) No cell is allowed to be missing a color – all 36 cells need to be colored.

Problem 5. Dan knows four numbers that add up to 99. If you multiply the first number by 2, add 2 to the second number, divide the third number by 2, and subtract 2 from the last number, you get four identical numbers. What are Dan's numbers?

Problem 6. You are given 5 bags of coins. It is known that four of them contain lots of good coins that weigh 5 grams each. The fifth bag has fake coins that weigh 4 grams each but look exactly the same as good coins. You also have a scale that shows total weight of coins put on it. How can you determine which bag contains bad coins in just one trial on this scale?

Math-M-Addicts Group Y Entrance Exam 2016-2017

Student Name_________________________________

Instructions: Please write legibly and fully justify your answers. Points will be deducted for incomplete solutions. At the same time, it is OK to provide partial answers, as you may earn points for good ideas even if you do not have full solutions. Good luck!

Problem 1 (10 points): A and B are 3-digit numbers. The product of the three digits of number A is equal to 12, and so is the product of the digits of B. Find the largest possible value of $B - A$.

Problem 2 (10 points): Emily's walk-in closet contains 10 pairs of white shoes, 15 pairs of red shoes, and 20 pairs of black shoes. One day the lightbulb in the closet goes out, so Emily can no longer tell a right shoe from a left shoe, or the color of any shoe. What is the smallest number of shoes Emily needs to bring outside to be absolutely sure she has obtained a right shoe and a left shoe of the same color, though not necessarily from the same pair?

Problem 3 (10 points): Bumblebee Open is a tennis tournament that starts with 128 players. All of the players are paired up for round 1. The loser of each match is considered knocked out and leaves. The winners from round 1 are paired up for round 2. The losers of round 2 matches leave, while the winners are paired up for round 3, and so on until the final match is played. At the conclusion of the tournament, how many players have won an odd number of matches?

Problem 4 (10 points): Danny really likes mixing coffee and milk. The other day in Starbucks coffee shop, he asked for "coffee with room for milk" and received a cup which was only 75% full. Danny poured in some milk to fill the cup and mixed his coffee and milk thoroughly. After drinking a quarter of the cup, Danny filled it up again with milk and mixed thoroughly. Then the same thing happened again: he drank a quarter of the cup and filled the empty space with milk. Finally, Danny finished his drink. Did he drink more coffee or more milk? Explain your answer.

Problem 5 (10 points): Tom is celebrating his birthday today. His age is equal to the sum of the digits of the year he was born. How old might Tom be? Give all possible answers.

Problem 6 (10 points): The number 100 can be written as a sum of five positive integers (positive whole numbers) using each digit no more than once. For example, $100 = 90 + 1 + 2 + 3 + 4$ or $100 = 57 + 31 + 2 + 4 + 6$. Find a way to express 2016 as a sum of five positive integers using each digit no more than once.

Math-M-Addicts Entrance Exam Solutions
Group Y 2014 – 2015

Problem 1. Using numbers 5, 6, 7, 8 and basic arithmetic operations (i.e., +, -, *, /, and parentheses), generate 24. Each number has to be used exactly once and combining numbers (e.g., 78 - 56) is not allowed.

Solution: We note that 24 = 12 * 2 = 8 * 3 = 6 * 4, which is reflected in our solutions: $\boxed{24 = 6 * (7 + 5 - 8)}$, $\boxed{24 = 8 * 6 / (7 - 5)}$, $\boxed{24 = (7 + 5) * (8 - 6)}$.

Problem 2. Find all solutions to TAE + TEA = EAT; each letter corresponds to a digit; same letter corresponds to
the same digit and different letters correspond to different digits.

Solution 1: Consider the addition of the middle digits: A + E = A. It can only be possible if E = 0 and there was a carry of 0 or if E = 9 and there was a carry of 1 (since we are adding just two numbers, only carries of 0 and 1 are possible). Since EAT starts with E, E cannot be 0, therefore E = 9. Therefore, from looking at leading digits, T = 4, which leads to A = 5 from looking at the last digit. Answer: $\boxed{T = 4, A = 5, \text{ and } E = 9}$.

Solution 2: The statement of the problem is equivalent to

$$100T + 10A + E + 100T + 10E + A = 100E + 10A + T,$$

which simplifies to

$$199T + A = 89E$$

Since TAE + TEA is a 3-digit number, T = 1, 2, 3, or 4. Plugging in, we find that T = 1, T = 2, and T = 3 yield no solutions because 89E cannot come close enough to 199T, and T = 4 yields the solution $\boxed{T = 4, A = 5, \text{ and } E = 9}$.

Problem 3. Positive numbers p and $p^2 + 7$ are both prime. What are all possible values of p? Justify your answer.

Solution: We note that p cannot be odd. Indeed, if p is odd, then $p^2 + 7$ is even; since the only even prime is 2, we would need $p^2 = -5$, which is impossible. Therefore, p is even, which means p =2. Checking, $p^2 + 7 = 11$, which is prime. Answer: $\boxed{p = 2}$.

Problem 4. A scientist gathered four math students. They were lined up so that each one could see the ones in front of them but not behind them. Each student had a hat placed on their head. So the student in the back could see the hats of the three students in front, but the student in front could not see any hats. *"There is a red hat, a white hat, a blue hat, and a hat that is a duplicate of one of those colors,"* the scientist said. Starting with the one in the back, each student was asked what color hat they were wearing. They were all able to give correct answers without guessing! Which two students were wearing the two hats of the same color? Explain your reasoning.

Solution: The last student must have seen two hats of the same color to guess his color correctly for if
all three hats in front of him were different colors, there would be no way for him to determine the color of his hat. Now consider the next-to-last student. If the hats in front of him are of different colors, he does not know whether his color matches the color of the first or the second hat. Therefore, the hats in front of him also need to be of the same color. The last two students will know that their hats are of the same, unmentioned, color. Hence, the two students whose hats are the same color are the first two.

<u>Problem 5</u>. If 1 + 2 + 3 + 4 + ... + 300 = 45,150, compute 2 + 4 + 6 + 8 + ... + 300 without using a calculator. Explain your reasoning.

Solution 1: Let 2 + 4 + 6 + 8 + ... + 300 = X. Then,

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots + 300 = 45{,}150$$
$$(2-1) + 2 + (4-1) + 4 + (6-1) + 6 + \cdots + (300-1) + 300 = 45{,}150$$

Rearranging the terms, we get:

$$(2 + 4 + 6 + \cdots + 300) + (2 + 4 + 6 + \cdots + 300) - 1 - 1 - 1 \ldots - 1 = 45{,}150$$

Noting that there are exactly 150 -1's and recalling our definition of X, we obtain:

$$2X - 150 = 45{,}150$$
$$X = 22{,}650$$

Answer: $\boxed{22{,}650}$.

Solution 2: We ignore the help we are given and compute 2 + 4 + 6 + 8 + ... + 300 directly. Matching up the first number with the last number, the second number with the next-to-last number, etc., we get 302 + 302 + … + 302, 75 times since we have 150 numbers. Answer = 302 * 75 = $\boxed{22{,}650}$. The answer makes sense since it is about half of 45,150.

<u>Problem 6</u>. All numbers between 1 and 999 were written out into one very large number. How many 7's were used? Explain your reasoning.

Solution 1: We will go ahead and count directly. The last digit is 7 in 7, 17, 27, …, 997 – one number in ten, which is 100 numbers. The first digit is 7 in 700 – 799 – also 100 numbers. Now, for the middle 7. It is in 10 numbers between 0 and 100 (70 – 79), in 10 numbers between 100 and 199 (170 – 179), etc., up until occurring 10 times as the middle digit between 900 and 999 (970 – 979), for a grand total of also 100. Answer: 100 + 100 + 100 = $\boxed{300}$.

Solution 2: Let's change the problem – instead of writing numbers from 1 to 999, we write numbers from 0 to 999. Clearly the answer does not change. Now, let's change the problem again – instead of writing out a number, we will also write out the leading 0's so as to complete a 3-digit number. That is, we will write 000, 001, 002, 003, …, 010, 011, …, 099, 100, 101, …, 999. Now, note that we wrote every combination of 3 digits exactly once! So each digit is repeated the same number of times, by symmetry. Since there are 1,000 numbers, we wrote 3,000 digits, and therefore we wrote exactly $\boxed{300}$ 7's (as well as 300 1's, 300 2's, etc.).

Math-M-Addicts Entrance Exam Solutions
Group Y 2015 – 2016

Problem 1. A book has 120 pages numbered 1,2,3,…, 120. How many times is the digit “9” used in page numbers?

Solution: We compute the answer directly. Between 1 and 100, “9” appears as the units digits in 10 times and as the tens digit 10 times. After 100, it appears in 109 and 119. Answer = 20 + 2 = $\boxed{22}$.

Problem 2. A box contains 100 balls. They are colored red, white and blue. Emily is picking the balls from the box without looking. What is the smallest number of balls she can take to be absolutely sure there would be 20 balls of the same color among the ones picked?

Solution: If Emily picks 57 balls, she could be unlucky and have 19 balls of each of the 3 colors. However, if she picks 58 balls, she needs to have at least 20 balls of at least one color (otherwise, she has at most 19 of each color, which would be at most 57 in total, which would be a contradiction). Answer: $\boxed{58}$.

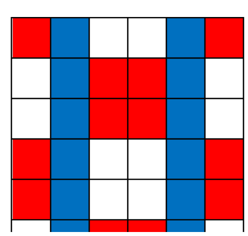
Problem 3. There are two water bottles; one of them can hold 8 quarts of water and the other can hold 5 quarts. You are on the shore of a lake. How can you use these bottles to get exactly 7 quarts of water in the bigger bottle?

Solution: The solution consists of successfully filling the 5-quart bottle, emptying it into the 8-quart bottle till the latter gets full, emptying the 8-quart bottle into the lake when it is full, and pouring the contents of the 5-quart bottle into the 8-quart bottle. Here is a table of what we will have in each bottle after each step:

	Step 1	**Step 2**	**Step 3**	**Step 4**	**Step 5**	**Step 6**	**Step 7**
5-qt bottle	5	5	2	2	0	5	0
8-qt bottle	0	5	8	0	2	2	7

Problem 4. Show how the 1x1 cells of a square 6x6 can be colored in 3 colors so that each cell shares a side with exactly one cell of the other two colors. (So if the colors are red, white and blue, and you have a white cell somewhere, it must have exactly one blue and one red neighbor.) No cell is allowed to be missing a color – all 36 cells need to be colored.

Solution: We enclose one such coloring using the colors of red, white, and blue; a good way of attempting the problem is to start from the



corner and to add colors, making decisions in line with the restrictions of the problem.

Problem 5. Dan knows four numbers that add up to 99. If you multiply the first number by 2, add 2 to the second number, divide the third number by 2, and subtract 2 from the last number, you get four identical numbers. What are Dan's numbers?

Solution: Let the 4 identical numbers in the end all equal to x. Then Dan's numbers were $x/2$, $x-2$, $2x$, and $x+2$, in that order. Adding them up, one obtains $4.5x = 99$, yielding $x = 22$. Therefore, Dan's numbers were $\boxed{\text{11, 20, 44, and 24}}$.

(Note: the same answer can be obtained using bars rather than writing an equation.)

Problem 6. You are given 5 bags of coins. It is known that four of them contain lots of good coins that weigh 5 grams each. The fifth bag has fake coins that weigh 4 grams each but look exactly the same as good coins. You also have a scale that shows total weight of coins put on it. How can you determine which bag contains bad coins in just one trial on this scale?

Solution: One solution is to pick 1 coin from bag one, 2 coins from bag two, and so on, and put all those coins on the scale. If all coins were authentic, the total weight of the pile would have been 5 * (1 + 2 + 3 + 4 + 5) = 75 grams, but because one bag has lighter coins, the total weight will be lower. One can see that if the pile weighs 74 grams, the first bag contains fake coins. Similarly, if the pile weighs 73 grams, the second bag contains fake coins, and so on. (In general, if the bag weighs 75 – k grams, then bag k has fake coins.)

Math-M-Addicts Group Y Entrance Exam Solutions:
Group Y 2016 – 2017

Problem 1 (10 points): A and B are 3-digit numbers. The product of the three digits of number A is equal to 12, and so is the product of the digits of B. Find the largest possible value of $B - A$.
Solution: To make $B - A$ as large as possible, we make B as large as possible and A as small as possible. To maximize B, we maximize its first digit to make it 6, the largest 1-digit divisor of 12. The remaining two digits multiply out to 2, so they need to be 2 and 1. Therefore, B = 621 since 621 > 612. Now, to come up with the smallest number, we reverse the order of digits of B or, alternatively, we proceed with a similar argument – the smallest first digit is 1, and then the smallest second digit is 2, meaning the last digit is 6. Therefore A = 126 and the answer is 621 – 126 = $\boxed{495}$.

Problem 2 (10 points): Emily's walk-in closet contains 10 pairs of white shoes, 15 pairs of red shoes, and 20 pairs of black shoes. One day the lightbulb in the closet goes out, so Emily can no longer tell a right shoe from a left shoe, or the color of any shoe. What is the smallest number of shoes Emily needs to bring outside to be absolutely sure she has obtained a right shoe and a left shoe of the same color, though not necessarily from the same pair?
Solution: We claim the answer is $\boxed{\text{46 shoes}}$. Indeed, 45 shoes are insufficient because they could all have been left shoes – 10 left white shoes, 15 left red shoes, and 20 left black shoes. However, if there are 46 shoes, then there are at least 11 white shoes, or at least 16 red shoes, or at least 21 black shoes. Without loss of generality, assume there are at least 16 red shoes. Since there are only 15 pairs of red shoes, we must have at least one left red shoe and at least one right red shoe, proving that 46 shoes are indeed sufficient.

Problem 3 (10 points): Bumblebee Open is a tennis tournament that starts with 128 players. All of the players are paired up for round 1. The loser of each match is considered knocked out and leaves. The winners from round 1 are paired up for round 2. The losers of round 2 matches leave, while the winners are paired up for round 3, and so on until the final match is played. At the conclusion of the tournament, how many players have won an odd number of matches?
Solution: People knocked out in the first round won 0 matches, those knocked out in 2nd round won 1 match, those knocked out in 3rd round won 2 matches, etc. Overall, the answer is the sum of the underlined terms: 64 (lost in 1st round), <u>32</u> (lost in 2nd round), 16 (lost in 3rd round), <u>8</u> (lost in 4th round), 4, <u>2</u>, 1, <u>1</u>, which is $\boxed{43}$.

Problem 4 (10 points): Danny really likes mixing coffee and milk. The other day in Starbucks coffee shop, he asked for "coffee with room for milk" and received a cup which was only 75% full. Danny poured in some milk to fill the cup and mixed his coffee and milk thoroughly. After drinking a quarter of the cup, Danny filled it up again with milk and mixed thoroughly. Then the same thing happened again: he drank a quarter of the cup and filled the empty space with milk. Finally, Danny finished his drink. Did he drink more coffee or more milk? Explain your answer.
Solution: The only coffee that Danny drank was the initial 75% of the cup. As for milk, he added it three times, each time adding 25% of the cup, for a grand total of also 75% of the cup. Therefore, he drank $\boxed{\text{the same amount of coffee and milk}}$.

Problem 5 (10 points): Tom is celebrating his birthday today. His age is equal to the sum of the digits of the year he was born. How old might Tom be? Give all possible answers.

Solution: Assume Tom was born in the 21st century, in the year $\overline{20xy}$ (the bar above indicates that these are digits of a 4-digit number as opposed to the product of 20, x, and y). Given the current year is 2016, his age today is $16 - 10x - y$, which needs to be equal to $2 + x + y$, meaning that $14 = 11x + 2y$. Since $\overline{20xy}$ needs to be less than 2016, we have that $x = 0$ or $x = 1$, leading to the unique solution of $x = 0$ and $y = 7$, meaning that Tom turned 9.
Now, assume Tom was born in the 20th century, in the year $\overline{19xy}$. Then his age in 2016 would be $116 - 10x - y$, which needs to be equal to $10 + x + y$, meaning that $106 = 11x + 2y$. Since $y < 10$, we have a unique solution of $x = 8$ and $y = 9$, meaning that Tom turned 27.
Lastly, there are no solutions if Tom was born in the 19th century or before because then his age in 2016 would be at least $216 - 10x - y$, which is going to be greater than $9 + x + y$ even if $x = y = 9$ – and the sum of the digits is at most $9 + x + y$.
Answer: $\boxed{9}$ or $\boxed{27}$.

Problem 6 (10 points): The number 100 can be written as a sum of five positive integers (positive whole numbers) using each digit no more than once. For example, $100 = 90 + 1 + 2 + 3 + 4$ or $100 = 57 + 31 + 2 + 4 + 6$. Find a way to express 2016 as a sum of five positive integers using each digit no more than once.
Solution: There are many possible answers; here is one of them: $\boxed{2016 = 1978 + 26 + 5 + 3 + 4}$. One thing to note is that 2016 is divisible by 9 since the sum of its digits is divisible by 9. Since each number leaves the same remainder as the sum of its digits when divided by 9 – and because the remainder of the sum is the sum of the remainders, it means that the sum of used digits is divisible by 9. Therefore, the correct solution will either have no digits missing, or 0 missing, or 9 missing.