Group A Math-M-Addicts Entrance Exam 2015 – 2016

Student Name ____Grade in School ____________________

School Name ______________________________Email ____

Instructions: A perfect solution to each problem below is worth 10 points. A score of 40 or more will guarantee admission to group A, though depending on the submitted tests, the cutoff for admission may be lower. Students must work on the problems on their own and not receive any help; students suspected of cheating will be disqualified. A reminder: this is a proof-based test for a proof-based class; please fully justify all your answers.

Solutions can be submitted in one of two ways:
(1) Faxed to (212) 845 - 1695
(2) Emailed to breydo@gmail.com

The submission deadline is Monday, September 14, 2015.

Good luck!

Problem 1. Let x and y be positive real numbers. Prove that $(x+y)(1+xy) \le (1+x^2)(1+y^2)$

Problem 2. Suppose that real numbers α, β and γ satisfy

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha * \tan\beta * \tan\gamma .$$

Find, with proof, all possible values of $\alpha + \beta + \gamma$.

Problem 3. A town council has 30 members. Each member of the council has exactly 6 enemies and 23 friends among other members. In how many ways can one select a commission of 3 councilmen so that either all 3 are friends with each other or all 3 are enemies of each other? (For the purposes of this problem, friendships/animosities are mutual relationships, i.e. if A is a friend of B then B is also a friend of A, and similarly for enemies).

Problem 4. In triangle ABC, CH is the altitude to side AB. (for the avoidance of doubt, H lies on a line AB and CH $\perp$ AB). $\angle BAC = 75°$, and $|CH| = \frac{1}{2}|AB|$. Find, with proof, all possible values of $\angle ABC$.

Problem 5. Find, with proof, all positive integers n such that $2^n - 1$ is a multiple of n (i.e. $n \mid 2^n - 1$) .

Problem 6. 120 1x1 squares are placed inside a 20x25 rectangle in an arbitrary way (in particular, squares may overlap with each other and their sides do not have to be parallel to the sides of the big rectangle). Prove that no matter how these 120 squares are placed, one can always put a circle of diameter 1 fully inside this rectangle so that it does not intersect any of the squares.