Student Name: **School:** **Grade:**

Math-M-Addicts Group S-L Entrance Exam (SAMPLE)

Instructions: The Group S exam consists of questions 1-6, and the Group L exam consists of questions 4-9. If you know which exam you would like to take, solve only the questions for that exam. If you would like to test for both exams, attempt all of the questions! Admission to each Group will be determined separately, and will be based solely on the questions for that Group.

MMA Only:
S: ___
L: ___

Please write legibly and fully justify your answers to receive partial credit.

S **Problem 1:** Numbers 1, 2, 3, …, 20 were written on a whiteboard. Erika erased two of the numbers, so that the remaining eighteen numbers add up to 183. What is the largest possible value of the difference between the two erased numbers?

S **Problem 2:** A box contains four different pairs of socks: one red pair, one blue pair, one purple pair and one green pair. Joe took 4 random socks out of the box. What is the probability that all the socks that Joe took are of different colors (i.e. one red, one blue, one purple and one green)?

S **Problem 3:** A positive integer N is a multiple of 51. When $\frac{N}{3}$ is divided by $\frac{N}{17}$ with remainder, that remainder is equal to 20 (note that both $\frac{N}{3}$ and $\frac{N}{17}$ are integers). Show that there is a unique natural number N satisfying these conditions and find it.

S/L **Problem 4:** There are a total of 47 students in Math-m-addicts S groups. Is it possible that each of them has exactly 1, 5, or 9 friends amongst those 47? (In this problem, we treat friendships as symmetric; i.e. if Bob is Peter's friend than Peter is also Bob's friend).

S/L **Problem 5:** In a number N = 17?552?8 the two question marks stand for unknown digits that do not have to be the same. It is known that N is a multiple of 72. Find all possible values of N.

S/L **Problem 6:** Polling company Opinions & Co talked to 50 residents of a town that held an election of a mayor. Each resident voted for precisely one candidate. Prove that Opinions & Co can find either 8 residents who voted for the same candidate or 8 residents who voted for 8 distinct candidates.

L **Problem 7**: A teacher wrote out the first few odd numbers 1, 3, 5…. on a whiteboard. Mary erased one of the numbers. The remaining numbers add up to 700. Which number was erased by Mary? Find all possibilities and prove that there are no others.

L **Problem 8:** Department of Transportation has 9 parking spots, arranged in a straight line. A parked truck occupies three adjacent spots, a parked bus occupies two adjacent spots while a parked car occupies one spot. How many ways are there to park a green truck, a yellow bus, a red bus and a blue car into those 9 spots?

L **Problem 9:** 5 rooks are placed on a 5x5 chessboard in such a way that no two rooks attack each other. Prove that there exists a 2x2 square on the chessboard which does not contain any of the rooks.

Math-M-Addicts Group S-L Entrance Exam (SAMPLE) Solutions

$\boxed{\text{S}}$ **Problem 1:** Numbers 1, 2, 3, …, 20 were written on a whiteboard. Erika erased two of the numbers, so that the remaining eighteen numbers add up to 183. What is the largest possible value of the difference between the two erased numbers?

Solution: The sum of the numbers as originally written is 1 + 2 + … + 20 = $\frac{20*21}{2}$ = 210. Therefore, the sum of the two erased numbers is 210 – 183 = 27. The largest difference is obtained when the numbers are as far from each other as possible, namely 20 and 7. Answer: 20 – 7 = $\boxed{13}$.

$\boxed{\text{S}}$ **Problem 2:** A box contains four different pairs of socks: one red pair, one blue pair, one purple pair and one green pair. Joe took 4 random socks out of the box. What is the probability that all the socks that Joe took are of different colors (i.e. one red, one blue, one purple and one green)?

Solution 1: The probability that the color of the second sock Joe chose is different from the color of the first sock he chose is 6/7. The probability the color of the third sock is different from the color the first two socks is 4/6. The probability the fourth sock is of a different color than the first three socks is 2/5. Since these are independent events, we multiply their probabilities to get: $\frac{6*4*2}{7*6*5} = \frac{8}{35}$. Answer: $\boxed{8/35}$.

Solution 2: There are $\binom{8}{4}$ ways of choosing the 4 socks. That's the denominator. As for the numerator, there are $\binom{2}{1}$ ways of choosing a sock of each color, and they are all independent. Therefore, the answer is $\frac{\binom{2}{1}*\binom{2}{1}*\binom{2}{1}*\binom{2}{1}}{\binom{8}{4}} = \frac{16*24}{8*7*6*5} = \frac{8}{35}$. Answer: $\boxed{8/35}$.

$\boxed{\text{S}}$ **Problem 3:** A positive integer N is a multiple of 51. When $\frac{N}{3}$ is divided by $\frac{N}{17}$ with remainder, that remainder is equal to 20 (note that both $\frac{N}{3}$ and $\frac{N}{17}$ are integers). Show that there is a unique natural number N satisfying these conditions and find it.

Solution: Let $N = 51k$, where k is a positive integer. Then $\frac{N}{3} = 17k$ and $\frac{N}{17} = 3k$. Dividing $17k$ by $3k$, one gets 5 with a remainder of $2k$, which needs to be equal to 20. Therefore, the only possibility is $k = 10$. Now, we check that $\boxed{510}$ actually works: 170 divided by 30 indeed leaves are remainder of 20. This works!

$\boxed{\text{S/L}}$ **Problem 4:** There are a total of 47 students in Math-m-addicts S groups. Is it possible that each of them has exactly 1, 5, or 9 friends amongst those 47? (In this problem, we treat friendships as symmetric; i.e. if Bob is Peter's friend than Peter is also Bob's friend).

Solution: It is $\boxed{\text{impossible}}$. We will prove it by contradiction. Assume it is possible, and let us count the sum of friendships across the 47 Math-m-addicts S group students. On one hand, we should get an even number, because we will have counted each friendship twice. On the other hand, we will get an odd number because total number of students is odd, and each student has an odd number of friends. So we have a contradiction, therefore it is impossible, QED.

$\boxed{\text{S/L}}$ **Problem 5:** In a number N = 17?552?8 the two question marks stand for unknown digits that do not have to be the same. It is known that N is a multiple of 72. Find all possible values of N.

Solution: Call the number 17A552B8. In order to be divisible by 72, it needs to be divisible by 8 and by 9. To be divisible by 8, the number made from the last 3 digits needs to be divisible by 8; as 208 is divisible by 8, the possibilities for B are 0, 4, and 8. To be divisible by 9, the sum of the digits needs to be divisible by 9, which means that A + B = 8 or A + B = 17. This yields the following four answers:

- N = 17855208
- N = 17455248
- N = 17055288
- N = 17955288

$\boxed{\text{S/L}}$ **Problem 6:** Polling company Opinions & Co talked to 50 residents of a town that held an election of a mayor. Each resident voted for precisely one candidate. Prove that Opinions & Co can find either 8 residents who voted for the same candidate or 8 residents who voted for 8 distinct candidates.

<u>Solution:</u> Assume the contrary. Then, no more than 7 candidates received votes – otherwise, we could find 8 residents who voted for 8 distinct candidates. Also, no more than 7 residents voted for each candidate – otherwise we could find a candidate with 8 or more votes. However, this means there were at most 7 * 7 = 49 votes, but as 50 residents voted, we have a contradiction. Therefore, Opinions & Co can find either 8 residents who voted for the same candidate or 8 residents who voted for 8 distinct candidates, QED.

$\boxed{\text{L}}$ **Problem 7**: A teacher wrote out the first few odd numbers 1, 3, 5…. on a whiteboard. Mary erased one of the numbers. The remaining numbers add up to 700. Which number was erased by Mary? Find all possibilities and prove that there are no others.

<u>Solution 1:</u> Let N be the number of odd numbers written by the teacher. Recall that the sum of the first N odd numbers is equal to N^2 (this can be easily proven by induction). After Mary deleted one of the numbers, there are now (N-1) numbers on the board and the minimum their sum can be is $N^2 – (2N-1)=(N-1)^2$. (The biggest number written by the teacher is Nth odd number, which is (2N-1)). Therefore, the sum of the remaining numbers on the board is between $(N-1)^2$ and N^2. Since 700 is between 26^2 and 27^2, we conclude that N = 27, which means that the missing number is $27^2 – 700 = 29$. Answer: $\boxed{29}$.

<u>Solution 2</u>: Recall that the sum of the first N odd numbers is equal to N^2 (this can be easily proven by induction), therefore the sum of the numbers before a number was erased is a perfect square above 700. If it is $27^2 = 729$, then it means 29 was erased. It cannot be 28^2, because then the erased number would need to be even, which is impossible. If it is $29^2 = 841$, it is also impossible, because then the erased number would need to be 141, whereas the largest number she can erase is 57. It cannot be larger than 29^2 because each time the largest term only grows by two, whereas the square grows by much more than that. Answer: $\boxed{29}$.

$\boxed{\text{L}}$ **Problem 8:** Department of Transportation has 9 parking spots, arranged in a straight line. A parked truck occupies three adjacent spots, a parked bus occupies two adjacent spots while a parked car occupies one spot. How many ways are there to park a green truck, a yellow bus, a red bus and a blue car into those 9 spots?

<u>Solution</u>: When a truck (3 spots), a yellow bus (2 spots), a red bus (2 spots), a car (1 spot) all park, there will be 1 spot left over. So the question becomes: how many distinct ways are there to arrange a “3”, “2a”, “2b”, “1a”, and “1b” in a line, which is 5! = 120. Answer: $\boxed{120}$.

$\boxed{\text{L}}$ **Problem 9:** 5 rooks are placed on a 5x5 chessboard in such a way that no two rooks attack each other. Prove that there exists a 2x2 square on the chessboard which does not contain any of the rooks.

<u>Solution</u>: Assume the contrary, that it is possible to place the 5 rooks in such a way that each 2x2 square contains a rook. First, since the rooks don’t attack each other, there can only be at most 1 rook in each column and row.

Now, let’s consider the lowest 2x2 square. If a rook is in the corner (red square; see diagram on the next page), then the two 2x2 squares above, marked on the diagram, can only have 1 rook total. This means that at least one of them is rook-free, so we have a contradiction. If the rook is in the green square, exactly the same argument applies. If the rook is in the blue square, the same argument applies as well, but now we look at two 2x2 squares to the right as opposed to above (not labeled on the diagram).

Lastly, if the rook is in the innermost (yellow) square, we look at the topmost 2x2 square instead. A rook cannot be in its innermost square because then the two rooks would be attacking each other, therefore it is in one of the other three squares, and the same argument applies. Therefore, we have a contradiction, which means that there exists a 2x2 square on the chessboard which does not contain any of the rooks, QED.