MMA Sample Entrance Exam S Solutions

Problem 1. Two prime numbers were written on a whiteboard. Each of these numbers was increased by one. This caused their product to increase by exactly 100. Which numbers were originally written on the board?

Solution 1. Let's say that P = the product of the two prime numbers, and A = product of the two prime numbers after adding 1. The problem tells us that A - P = 100.

If both prime numbers are odd, then P, their product, must be odd. One plus each odd number is even. Their product, A, is also even. A-P must be odd.

□

Solution 2. Let the prime numbers be p and q. Then

$$(p+1)(q+1) - pq = 100 \implies p+q+1 = 100 \implies p+q = 99.$$

So one of p, q must be odd and the other even. There is only one even prime, 2, so the numbers are $\boxed{\text{2 and 97}}$. □

Problem 2. There are six trails leading to the top of Mount Math. Sinai makes two hikes up and down the mountain, one in the morning and one in the afternoon. On each hike, she uses a different trail to descend than the trail she used to ascend, though the same trail can be used in both the morning and the afternoon hikes. In how many ways can Sinai plan her hikes?

Solution. For each hike, Sinai has 6 choices for the trail she takes up the mountain and 5 for the trail she takes down for 30 total; then for both hikes, she has $30 \cdot 30 = \boxed{900}$ total ways to plan. □

Problem 3. Each cell of a 3×3 square contains a number. The sum of numbers in each row equals 6. The sum of numbers in each column also equals 6. The sum of numbers in any 2×2 square equals 7. Determine, with proof, the number written in the central square.

Solution. Label the squares as follows:

a	b	c
d	e	f
g	h	i

Then $a+b+c+d+e+f = 12$ and $a+b+d+e = 7$, so $c+f = 5$. Therefore, $b+e = 2$ and $h = 4$, $i = 1$. A similar analysis on other pairs of rows and columns yields that $a = c = g = i = 1$ and $b = d = f = h = 4$. Then $e = -2$. □

Problem 4. Place numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 around a circle so that sum of any two numbers next to each other is not divisible by 3, 5, or 7.

Solution. For each number, we can check which numbers it can be adjacent to:

1	3, 7
2	6, 9
3	1, 5, 8
4	7, 9
5	3, 6, 8
6	2, 5, 7
7	1, 4, 6
8	3, 5, 9
9	2, 4, 8

Because the numbers are around a circle, every number is adjacent to exactly two others. Then, we see that 1 must be next to 3 and 7; that 4 must be next to 7 and 9; and that 2 must be next to 9 and 6. Then the numbers next to 6 must be 2 and 5, and the numbers next to 5 are 8 and 6. So one way to place the numbers is 1, 7, 4, 9, 2, 6, 5, 8, 3 clockwise around the circle, and it is easy to verify that this works. □

Problem 5. How many two-digit numbers give a perfect square when added to its "mirror" (the two-digit number written with its digits in reverse order)?

Solution. Represent a two-digit number by $\overline{AB}$ with A and B digits from 1 to 9 (inclusive) and $\overline{AB} = 10A + B$. Then the mirror of this number is $\overline{BA} = 10B + A$ and their sum is $11A + 11B$, or $11(A + B)$. For this to be a perfect square, $A + B = 11 \cdot n^2$ for some integer n, but because A, B are digits, $n = 1$. There are then $\boxed{8}$ pairs of digits which satisfy this, corresponding to 29, 38, 47, 56, 65, 74, 83, and 29. □

Problem 6. A carpenter went to the store and bought 10 planks of wood. Each plank has a length that is a whole number of centimeters. The longest plank has a length of exactly 54 centimeters. Prove that there exist three planks that can be arranged to form a triangle.

Solution. Suppose there do not exist three planks that can be arranged to form a triangle. Consider the minimal lengths of shortest, second shortest, etc. planks. The shortest plank has minimal length 1, as does the second shortest. The next-shortest plank has minimal length 2 (otherwise, we could make a triangle out of 3 planks all of length 1). Likewise, the next-shortest plank has minimal length $3 = 2 + 1$, and the next-shortest has minimal length $5 = 3 + 2$, and so on. Repeating this (or noticing that the planks have lengths the Fibonacci numbers), the 10th plank has minimal length 55, but the longest plank is 54 centimeters, contradiction. □