Solutions, by Vidur Jasuja

Problem 1. Firstly, the left hand side of the equation must be even, and so at least one of x, y, z is even. Suppose without loss of generality x is even. Let $x = 2x_0$. Then

$$4x_0^2 + y^2 + z^2 = 4x_0yz.$$

This means $y^2 + z^2$ is divisible by 4, meaning both y, z are even as well. Let $y = 2y_0, z = 2z_0$. This gives us that

$$4x_0^2 + 4y_0^2 + 4z_0^2 = 16x_0y_0z_0,$$

or

$$x_0^2 + y_0^2 + z_0^2 = 4x_0y_0z_0.$$

Repeating this process again, it can be seen that all of x_0, y_0, z_0 are even, and if $2x_1 = x_0, 2y_1 = y_0, 2z_1 = z_0$, then

$$x_1^2 + y_1^2 + z_1^2 = 8x_1y_1z_1.$$

In fact, this process can be repeated indefinitely, and so by the principle of infinite descent, no solutions can exist.

Problem 2. By Vieta's Formulas, $x_1 + x_2 = -p, x_1x_2 = \frac{1}{2p^2}$. So,

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = p^2 + \frac{1}{p^2}.$$

Now, repeating this,

$$x_1^4 + x_2^4 = (x_1^2 + x_2^2) - 2(x_1x_2)^2 = p^4 + 2 + \frac{1}{p^4} - \frac{1}{2p^4} = p^4 + 2 + \frac{1}{2p^4}.$$

By AM-GM,

$$p^4 + \frac{1}{2p^4} \geq 2\sqrt{\frac{1}{2}} = \sqrt{2},$$

so then the answer is $\sqrt{2} + 2$. which by setting p^4 and $\frac{1}{2p^4}$ equal is achieved at $p = 2^{-\frac{1}{8}}$.

Problem 3. Firstly, suppose lines AR and BT intersect at X, and lines DR and CT intersect at Y. Observe that $XRYT$ is a parallelogram due to our parallel conditions. The key claim is that $\triangle ABX$ is similar to $\triangle CYD$. It is clear that $\triangle AXT$ is similar to $\triangle TYD$. So,

$$\frac{AX}{YT} = \frac{TX}{DY}.$$

Analogously,

$$\frac{XR}{YC} = \frac{BX}{RY}.$$

Multiplying these two similarities, and also observing that $YT = XR$ and $TX = RY$,

$$\frac{AX}{CY} = \frac{BX}{DY}.$$

Now, since $\angle AXB = \angle TXR = \angle TYR = \angle CYD$, by SAS similarity, the claim is proven. Therefore, since $\angle BAX = \angle YCD$ and $AX \parallel YC$, it must be that $AB \parallel CD$, as desired.

Problem 4. Consider the sum of the reciprocals of the numbers on the board. The key claim is that this quantity is invariant. Indeed, if a and b are replaced by $\frac{ab}{a+b}$, note that

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{1}{\frac{ab}{a+b}}.$$

Thus, this quantity is invariant as desired. Therefore, the final number on the board has reciprocal

$$1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{512}=\frac{512+256+128+\cdots+1}{512}=\frac{1023}{512}.$$

Therefore, the final number on the board will always be $\frac{512}{1023}$, regardless of the order of operations.

Problem 5. Firstly, if there are an odd number of negative numbers among the x_i's, then their product is negative, and the conditions are all satisfied. Furthermore, if one of them is zero, the problem condition is also satisfied. So, now consider the case where 4 of the $x_i's$ are positive, and 2 are negative; this is symmetric with the case where 2 are positive and 4 are negative.

Suppose that x_1, x_2, x_3, x_4 are positive, and x_5, x_6 negative. Let $y_5=-x_5, y_6=-x_6$. Then $x_1+x_2+x_3+x_4=y_5+y_6=c$, for some constant c. Now, as a corollary of the AM-GM inequality (or Cauchy-Schwarz, or Jensen's), note that

$$x_1^2+x_2^2+x_3^2+x_4^2 \geq \frac{(x_1+x_2+x_3+x_4)^2}{4}=\frac{c^2}{4},$$

and

$$y_5^2+y_6^2 \geq \frac{(y_5+y_6)^2}{2}=\frac{c^2}{2}.$$

Therefore, summing these two inequalities,

$$6 \geq \frac{3c^2}{4} \rightarrow 2\sqrt{2} \geq c.$$

Now apply AM-GM.

$$\frac{x_1+x_2+x_3+x_4}{4} \geq \sqrt[4]{x_1x_2x_3x_4} \rightarrow \frac{\sqrt{2}}{2} \geq \sqrt[4]{x_1x_2x_3x_4} \rightarrow \frac{1}{4} \geq x_1x_2x_3x_4,$$

and

$$\frac{y_5+y_6}{2} \geq \sqrt{y_5y_6} \rightarrow \sqrt{2} \geq \sqrt{y_5y_6} \rightarrow 2 \geq y_5y_6 = x_5x_6.$$

Therefore,

$$x_1x_2x_3x_4x_5x_6 \leq \frac{1}{4}\cdot 2=\frac{1}{2},$$

as desired.

Problem 6. Suppose for the sake of contradiction V and W are airports such that they are 65 or more connections away. Consider vertex V. For a given positive integer i, let S_i denote the set of airports whose shortest travel to V involves exactly i flights. Now, note the following observations:

1. S_1 must contain at least 100 airports.
2. Each of $S_1, S_3, \ldots, S_{66}$ must contain at least one airport.
3. An airport in S_j can only be connected to airports in S_{j-1}, S_j, S_{j+1}. Indeed, if this airport is connected to an airport in S_k for $k \leq j-2$, then this airport would require less than j flights to get to V, a contradiction. If $k \geq j+2$, then that airport would require at most $j+1<k$ connections to get to V, a contradiction.

Now, combine these observations. Consider an airport in S_i; it is connected to at least 100 airports, and all of those must be in S_{i-1}, S_i, S_{i+1}. So, $|S_{i-1}|+|S_i|+|S_{i+1}| \geq 101$. Now,

$$S_1+(S_2+S_3+S_4)+(S_5+S_6+S_7)+\cdots(S_{62}+S_{63}+S_{64}) \geq 100+21\cdot 101=2221.$$

Therefore, there are at least 2222 airports among $V, S_1, S_2, S_3, \ldots, S_{64}$. This is evidently a contradiction; there are only 2016 airports. Therefore, a flight between any two airports must take at most 65 connections.

Solution 2. In a similar way, suppose there exist two airports V_0 and V_{66} that require sixty-five connections. Let the airports on the shortest path between these two be $V_1, V_2, V_3, \ldots, V_{65}$. Let S_k denote the set of V_{3k} and its neighbors. Then very similarly as before, $S_0, S_1, \ldots, S_{22}$ are disjoint, and each contain at least 101 airports. This means there are at least $23 \cdot 101 = 2323$ airports, a contradiction. In fact, $20 \cdot 101$ is already greater than 2016, so we could reduce the number of connections to 56 instead of 65.